Triangles as elements of algebraic solids

1. triangles

The triangles of three-dimensional Euclidean space are considered. We group them into classes of equivalent triangles; all triangles of the same **shape**, **centring** and **position** should belong to the same class. Triangles that are similar to each other have the "same shape"; a triangle can be "centred" in six ways, i.e. a corner can be highlighted in three ways and the sense of rotation can be selected in two ways.

(Fig. 1); triangles of the same "position" have parallel normals $\vec{a}_1 \times \vec{a}_2$ (Fig. 2).

When we refer to a "triangle" in the following, we mean a class of equivalent

Triangles in the sense described above. Each such triangle has a vector bipod as shown in Fig. 2: The vector $\vec{a}_1$ is spanned between the point A selected during centring and the "next point" B defined by the direction of rotation (also assumed), and the span vector $\vec{a}_2$ is located between A and the third point C of the triangle.

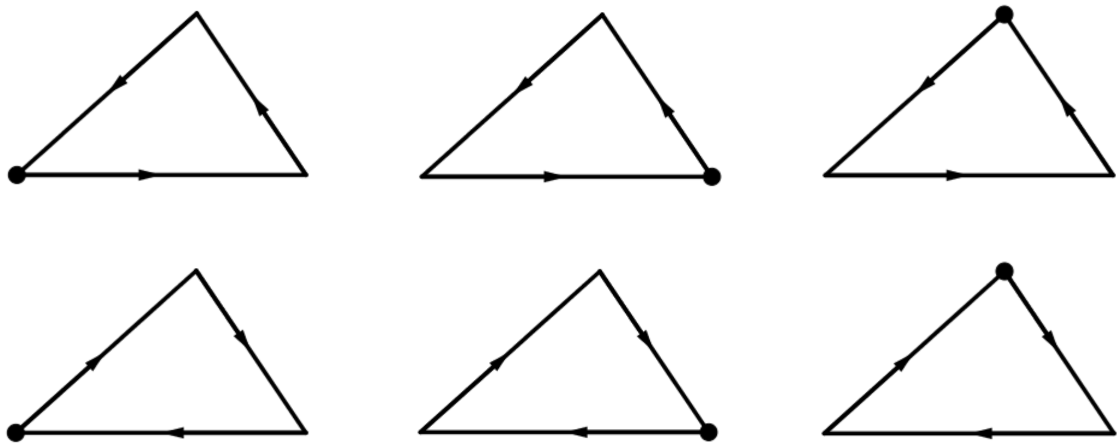


Fig. 1 The six ways to centre a triangle

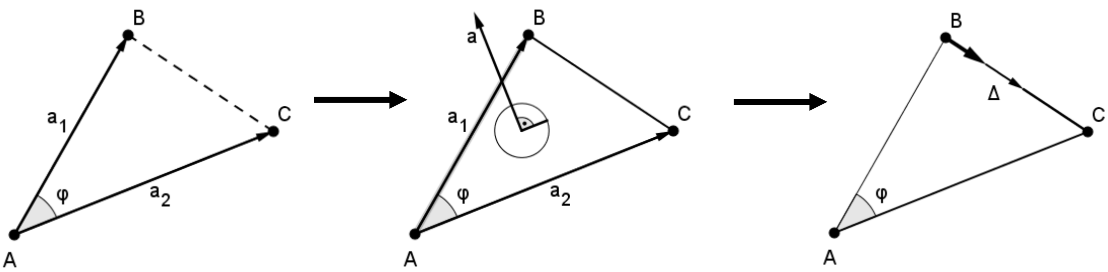


Fig. 2: The bipod associated with a triangle $\vec{a}_1, \vec{a}_2$,
the associated essential vector $\vec{a}$ and the associated number α

The essential vector $\vec{a}$ of the triangle is defined:

(1) $$\vec{a} := -\frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_2|^2} = \frac{\vec{a}_2 \times \vec{a}_1}{|\vec{a}_2|^2} \;, \quad (\,)\; \vec{a}_2 \neq \vec{n}$$

where $\vec{a}_1 \times \vec{a}_2$ is the outer product of two vectors and $\vec{n}$ is the zero vector.
The **essential number** α of the triangle is defined by

$$(2) \qquad \alpha := \frac{\vec{a}_1 \cdot \vec{a}_2}{|\vec{a}_2|^2}, \qquad (\,)\ \vec{a}_2 \neq \vec{n}$$

where $\vec{a}_1 \cdot \vec{a}_2$ is the inner product of the vectors $\vec{a}_1$ and $\vec{a}_2$
There is a $\vec{a}$ and a α for each of our triangles (with $\vec{a}_2 \neq \vec{n}$).
Conversely, a triangle is also assigned to each $\vec{a} \neq \vec{n}$ and any real α :

With the unit vector

$$(3) \qquad \frac{\vec{a}}{|\vec{a}|} = -\frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|} \qquad (\,)\ \vec{a} \neq \vec{n}$$

defines the position of the triangle and its direction of rotation.
And

$$(1^*) \qquad |\vec{a}| = \frac{|\vec{a}_1|}{|\vec{a}_2|} \sin\varphi$$

determined together with

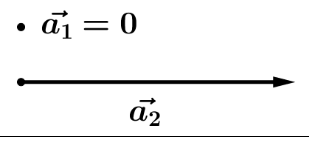
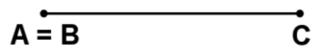
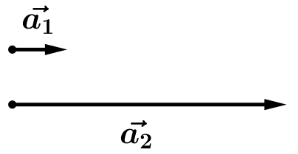
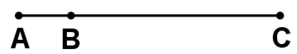
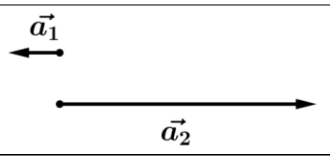
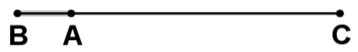
$$(2^*) \qquad \alpha = \frac{|\vec{a}_1|}{|\vec{a}_2|} \cos\varphi$$

the opening angle (with). $0 < \varphi < \pi$

It is $\quad \cot\varphi = \frac{\alpha}{|\vec{a}|}, \rho := \frac{|\vec{a}_1|}{|\vec{a}_2|} = \frac{|\vec{a}|}{\sin\varphi}$.

However, the opening angle φ and aspect ratio ρ determine the shape of the triangle.
Each of our "triangles" is therefore represented by a pair of $\Delta = (\alpha | \vec{a})$
consisting of a vector and a real number . α

We will include the case $\vec{a} = \vec{n}$ in the following considerations; here the interpretation is simpler and can be understood as a borderline case (degenerate case) of triangles.

	Vector bipod	degenerated triangle
$\alpha = 0$		
$\alpha > 0$		
$\alpha < 0$		

(Figure 3): The degenerate triangles ,$(\alpha, \vec{n})$ $\vec{a}_2 \neq \vec{n}$ are specified as required.

We assume an arbitrary $\vec{a}_2 \neq \vec{n}$. If $\alpha = 0$, we find , i.e. $|\vec{a}_1| = 0$ $\vec{a}_1 = \vec{n}$; if $|\vec{a}_1| \neq 0$, we would have $\varphi = \frac{\pi}{2}$ in (2*), which would lead to $|\vec{a}| \neq 0$ in (1*) and would be a contradiction to the assumption; with $\vec{a}_1 = \vec{n}$ (1) and (2) are also fulfilled.

In the case of $\alpha \neq 0$, (2*) $|\vec{a}_1| \neq 0$ and $\varphi \neq \frac{\pi}{2}$, which in turn yields (1*) $\varphi = 0$ or , i.e. $\varphi = \pi$ $\vec{a}_1 = \lambda \vec{a}_2$ $(\lambda \neq 0)$.

When inserted into (2*), the result is $\left.\begin{array}{l} \varphi = 0: \ \alpha = \ |\lambda > 0| \\ \varphi = \pi: \ \alpha = -|\lambda < 0| \end{array}\right\} \Rightarrow \vec{a}_1 = \alpha \vec{a}_2$

(1) and (2) are obviously fulfilled.

To summarise:
Each pair of $\Delta = (\alpha | \vec{a})$ (α is an arbitrary number, $\vec{a}$ is an arbitrary vector of R3) is now assigned one of our triangles, be it a real or a fake triangle.

2. the multiplication of triangles

Two triangles Δ_1 and Δ_2 can always be placed next to each other so that the 1st side of Δ_2 coincides with the 2nd side of Δ_1 . There is exactly one way to do this.
Two triangles joined in this way form a spatial corner (Figure 4).

With Δ_1 and Δ_2 the third page Δ_3 of this corner is clearly defined.
We write

(4) $\Delta_3 = \Delta_1 \circ \Delta_2$.

This two-digit ratio is recognised by

(5) $\Delta_3 = (\gamma | \vec{c}) = (\alpha\beta - \vec{a} \cdot \vec{b} | \alpha \vec{b} + \beta \vec{a} + \vec{a} \times \vec{b})$,

if $\Delta_1 = (\alpha | \vec{a})$ and $\Delta_2 = (\beta | \vec{b})$.

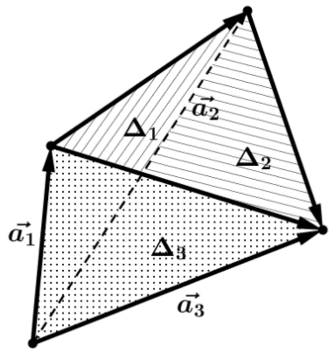


Figure 4 Geometric interpretation of $\Delta_1 \circ \Delta_2 = \Delta_3$

The proof is carried out using known development theorems for the inner and outer product.

a) $$\alpha\beta - \vec{a} \cdot \vec{b} = \frac{\vec{a}_1 \cdot \vec{a}_2}{|\vec{a}_2|^2} \cdot \frac{\vec{a}_2 \cdot \vec{a}_3}{|\vec{a}_3|^2} - \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_2|^2} \cdot \frac{\vec{a}_2 \times \vec{a}_3}{|\vec{a}_3|^2}$$

goes because of

$$(\vec{t} \times \vec{u}) \cdot (\vec{v} \times \vec{w}) = (\vec{t} \cdot \vec{v})(\vec{u} \cdot \vec{w}) - (\vec{u} \cdot \vec{v})(\vec{t} \cdot \vec{w})$$

about in

$$\alpha\beta - \vec{a} \cdot \vec{b} = \frac{1}{|\vec{a}_2|^2 \cdot |\vec{a}_3|^2} \left\{ (\vec{a}_1 \cdot \vec{a}_2)(\vec{a}_2 \cdot \vec{a}_3) - (\vec{a}_1 \cdot \vec{a}_2)(\vec{a}_2 \cdot \vec{a}_3) + (\vec{a}_2 \cdot \vec{a}_2)(\vec{a}_1 \cdot \vec{a}_3) \right\} = \frac{\vec{a}_1 \cdot \vec{a}_3}{|\vec{a}_3|^2} = \gamma$$

b) Firstly, using

$$\left(\vec{t}\times\vec{u}\right)\times\left(\vec{v}\times\vec{w}\right)=\left(\vec{t}\vec{u}\vec{w}\right)\vec{v}-\left(\vec{t}\vec{u}\vec{v}\right)\vec{w}\ ,$$

that

$$\alpha\vec{b}+\vec{a}\beta+\vec{a}\times\vec{b}=\frac{1}{\left|\vec{a}_2\right|^2\cdot\left|\vec{a}_3\right|^2}\left\{-\left(\vec{a}_1\cdot\vec{a}_2\right)\left(\vec{a}_2\times\vec{a}_3\right)-\left(\vec{a}_1\times\vec{a}_2\right)\left(\vec{a}_2\cdot\vec{a}_3\right)+\left(\vec{a}_1\times\vec{a}_2\right)\times\left(\vec{a}_2\times\vec{a}_3\right)\right\}=$$

$$=\frac{1}{\left|\vec{a}_2\right|^2\cdot\left|\vec{a}_3\right|^2}\left\{-\left(\vec{a}_1\cdot\vec{a}_2\right)\left(\vec{a}_2\times\vec{a}_3\right)-\left(\vec{a}_1\times\vec{a}_2\right)\left(\vec{a}_2\cdot\vec{a}_3\right)+\left(\vec{a}_1\vec{a}_2\vec{a}_3\right)\vec{a}_2\right\}$$

The abbreviation is set to

(7) $$\vec{x}:=\alpha\vec{b}+\vec{a}\beta+\vec{a}\times\vec{b}\ .$$

The inner multiplication with $\vec{a}_1,\vec{a}_2,\vec{a}_3$ leads to

$$\vec{x}\cdot\vec{a}_1=0\ ,$$

$$\vec{x}\cdot\vec{a}_3=0\ ,$$

$$\vec{x}\cdot\vec{a}_2=\frac{\left(\vec{a}_1\vec{a}_2\vec{a}_3\right)}{\left|\vec{a}_3\right|^2}\ .$$

The two equations are used to calculate

$$\vec{x}=\lambda\cdot\vec{a}_1\times\vec{a}_3\ ,$$

so that

$$\vec{x}\cdot\vec{a}_2=-\lambda\left(\vec{a}_1\vec{a}_2\vec{a}_3\right)=\frac{\left(\vec{a}_1\vec{a}_2\vec{a}_3\right)}{\left|\vec{a}_3\right|^2}\ .$$

At $\left(\vec{a}_1\vec{a}_2\vec{a}_3\right)\neq 0$ this means that

$$\lambda=-\frac{1}{\left|\vec{a}_3\right|^2}\ ,$$

so the assertion

$$\vec{x}=-\frac{\vec{a}_1\times\vec{a}_3}{\left|\vec{a}_3\right|^2}=\vec{c}\ .$$

The proof can also be provided at $\left(\vec{a}_1\vec{a}_2\vec{a}_3\right)=0$ by

$$\vec{a}_3=\mu_1\vec{a}_1+\mu_2\vec{a}_2$$

Introduced in (6).

3. the group property of the $\circ$ link.

It remains to show that the pairs form a group with regard to the multiplication explained for them in (5) if $(\mathbf{0}|\vec{u})$ is excluded. By inserting it into (5), you can immediately see that $(\mathbf{1}|\vec{u})$ is the neutral element. The validity of the associative law is illustrated in Fig. 5.

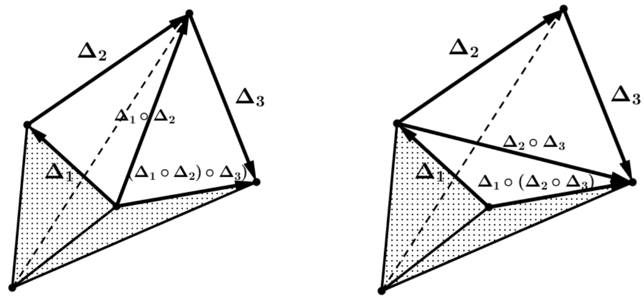


Fig. 5 Geometric proof of the validity of the associative law

$$(\Delta_1 \circ \Delta_2) \circ \Delta_3 = \Delta_1 \circ (\Delta_2 \circ \Delta_3)$$

It is calculated using the development theorem

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w}) \cdot \vec{v} - (\vec{u} \cdot \vec{v}) \cdot \vec{w}$$

The geometric meaning of the multiplication of the pairs makes it easy to construct the inverse element, because inverse triangles differ only in the direction of their position vector (Fig. 6).

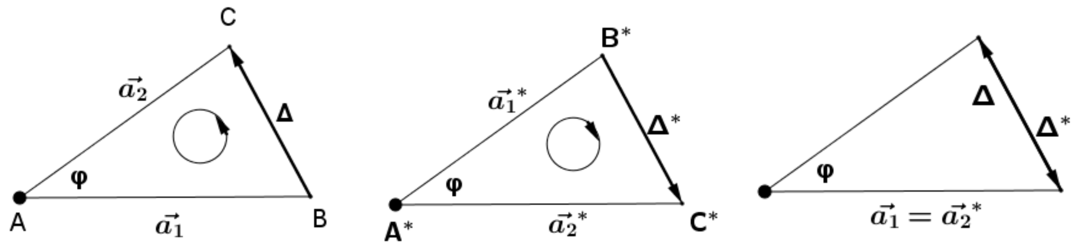


Fig. 6 Geometric interpretation of $\Delta \circ \Delta^* = (\mathbf{1}|\vec{n})$

If Δ has the aspect ratio ρ , then Δ^* has the ratio ρ^{-1} and because of $\varphi = \varphi^*$ follows

$$|\vec{a}^*| = \rho^{-1} \cdot \sin\varphi = |\vec{a}| \cdot \rho^{-2},$$

$$\vec{a}^* = \rho^{-1} \cdot \cos\varphi = \alpha \cdot \rho^{-2},$$

so that due to $\rho^2 = \alpha^2 + \vec{a}^2$

(8) $$(\alpha|\vec{a})^* = \left(\frac{\alpha}{\alpha^2 + \vec{a}^2} \middle| \frac{-\vec{a}}{\alpha^2 + \vec{a}^2} \right).$$

By recalculating, you confirm that this element Δ^* is inverse to . Δ

The calculation, like the visualisation, shows the invalidity of the communicative law.

4. the addition of triangles.
A second link is explained by

$$(\alpha|\vec{a}) * (\beta|\vec{b}) = \left(\alpha+\beta|\vec{a}+\vec{b}\right),$$

where + denotes the addition of both numbers and vectors. This connection does not allow a simple geometric interpretation on the basis of the triangles, but is algebraically obvious. It is easy to show that the triangles form an Abel group with respect to this connection. It also follows without any special prerequisites that both links are distributive:

$$\left(\Delta_1 * \Delta_2\right) \circ \Delta_3 = \Delta_1 \circ \Delta_3 * \Delta_2 \circ \Delta_3 .$$

The triangles therefore form an **oblique body** with respect to o and $*$.

5. coordinate display.
The vectors are related to an orthonormalised base:

$$\vec{a} = \alpha_1\vec{i} + \alpha_2\vec{j} + \alpha_3\vec{k} .$$

For α , the following is written α_0 , so that

$$(\alpha|\vec{a}) = \left(\alpha_0|\alpha_1\vec{i} + \alpha_2\vec{j} + \alpha_3\vec{k}\right).$$

The following follows from the oblique body axioms and the definition of $\circ$ and $*$

$$(\alpha|\vec{a}) = \left(\alpha_0|\vec{n}\right) * \left(0|\alpha_1\vec{i}\right) * \left(0|\alpha_2\vec{j}\right) * \left(0|\alpha_3\vec{k}\right) =$$
$$= \left(\alpha_0|\vec{n}\right) * \left(\alpha_1|\vec{n}\right) \circ \left(0|\vec{i}\right) * \left(\alpha_2|\vec{n}\right) \circ \left(0|\vec{j}\right) * \left(\alpha_3|\vec{n}\right) \circ \left(0|\vec{k}\right)$$

If you simply write + and $\cdot$ for $*$ and o; replace $\left(0|\vec{i}\right)$ with *i*; $\left(\alpha_0|\vec{n}\right)$ with α_0 etc., the following applies

(9) $$(\alpha|\vec{a}) = \alpha_0 + \alpha_1\vec{i} + \alpha_2\vec{j} + \alpha_3\vec{k} .$$

The products of the basis vectors follow from the definition of multiplication and the properties of the outer product:

	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	-1	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	-1	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	-1

Each element of this table can also be directly visualised using the
Gain by placing triangles next to each other (Fig. 7).

The oblique body of the triangles is isomorphic to the oblique body of the quaternions.
We also write a triangle (a quaternion) in "polar coordinates" in the form

(10) $$(\alpha|\vec{a}) = \rho\left(\cos\varphi + j\sin\varphi\right)$$

where ρ is the aspect ratio, φ the opening angle and *j* the position vector of the triangle with $j \circ j = -1$.

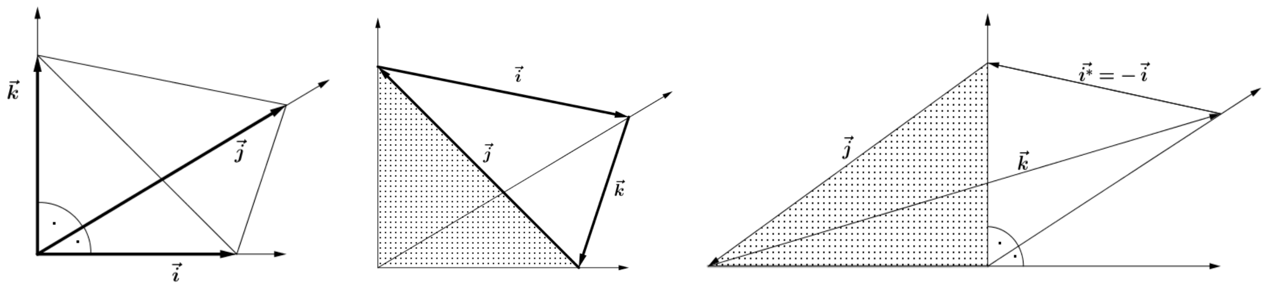


Fig. 7 Examples of triangle multiplication

$$(-\vec{i}) \circ \vec{j} = -\vec{k} \quad \Rightarrow \quad \vec{i} \circ \vec{j} = \vec{k} \qquad \vec{j} \circ \vec{i} = -\vec{k}$$

$$\vec{k} \circ (-\vec{i}) = -\vec{j} \quad \Rightarrow \quad \vec{k} \circ \vec{i} = \vec{j} \qquad \vec{i} \circ \vec{k} = -\vec{j}$$

$$\vec{j} \circ \vec{k} = \vec{i} \qquad \vec{k} \circ \vec{j} = -\vec{i}$$

6. lower body

It is easy to show that two triangles $(\alpha|\vec{a})$ and $(\beta|\vec{b})$ are interchangeable when multiplied if $\vec{a}$ and $\vec{b}$ are linearly dependent. Triangles whose position vectors kol are linear therefore form a commutative sub-body. Each of these bodies is isomorphic to the body of complex numbers. Triangles of such a sub-body can always be placed in a (Gaussian) plane.

7. summary

Taking more recent aspects into account, the historical connection between vector and quaternion arithmetic was brought back into focus. Didactically, it became clear that quaternions can be introduced geometrically as triangles, which are characterised algebraically by a pair consisting of number and vector. The geometric representation in a four-dimensional vector space is therefore unnecessary. Formally, the introduction is analogous to the introduction of complex numbers as pairs of real numbers. The skew-body axioms can be deduced by referring to arithmetic with real numbers and vectors. All axioms of multiplication can also be obtained graphically.

If one assumes that the concept of the vector is more descriptive than that of displacement, the author's experience may be plausible: The concept of the triangle used here is more descriptive than the concept of torsional extension. This is not only evident in the. This is evident not only in the treatment of quaternions, but also in the geometric introduction of complex numbers (Fig. 8).

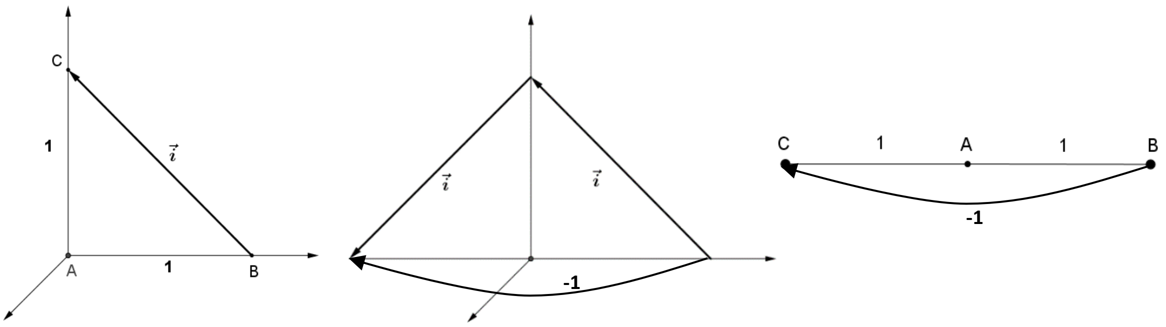


Figure 8: Explanation of $(0|\vec{i}) \circ (0|\vec{i}) = (-1|\vec{n})$

with the representatives , $\vec{a}_1 = \vec{k}$ $\vec{a}_2 = \vec{j}$ and $\vec{a}_1{}' = \vec{j}$, $\vec{a}_2{}' = -\vec{k}$

The pupils say: When interpreting the numbers as triangles, the non-real numbers are clearer than the real numbers, because these correspond to degenerate triangles with an opening angle of 0° or 180°.

The introduction of addition of whole numbers using the geometric model of vector addition is didactically recognised. This would correspond to the model of triangular multiplication to motivate the multiplication rules for whole numbers as shown in Fig. 9. If the first side of the "triangle" (-3) is placed against the second side of the "triangle" (-2), the result is the "triangle" (+6).

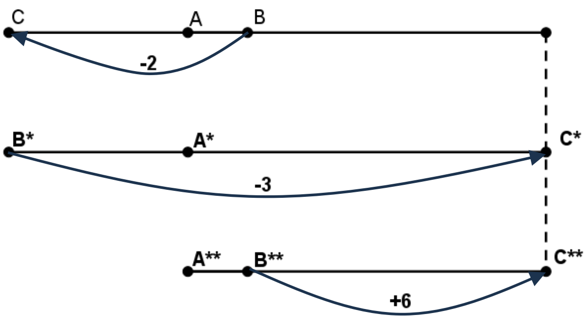


Fig. 9 Representation of $(-2|\vec{n}) \circ (-3|\vec{n}) = (+6|\vec{n})$